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# Evaluation of long-dated assets: The role of parameter uncertainty ☆

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ABSTRACT

Under expected utility, the uncertainty that affects the parameters of the random walk of consumption growth has no effect on the value of short-term claims and makes the term structure of risk-free rates decreasing. The term structure of aggregate risk premia is increasing when the uncertain cumulants of log consumption are independent. We apply these generic results to the case of an uncertain probability of catastrophes, and to the case of an uncertain trend or volatility of growth. Adding some persistence to unobservable shocks into our benchmark model, we show that the term structure of risk premia is hump-shaped.



## 1. Introduction

Do we do enough for the distant future? This question is implicit in many policy debates, from the fight against climate change to the speed of reduction of public deficits, investments in research and education, or the protection of the environment and of natural resources for example. The dual question is whether markets value assets with extra long-term cash-flows in the right way, thereby providing the efficient price signals to economic agents to invest for the long run. The discount rate used to evaluate investments is the key determinant of our individual and collective efforts in favor of the future. In relation to this question, an intense debate has emerged among environmental and climate economists since Weitzman (1998) about whether one should use different discount rates for different time horizons $t$.

A critical dimension to this problem is the deep uncertainties that surround the dynamics of economic growth. For example, a relatively small change in the permanent trend of growth has an immense impact on future consumption when projected over many decades or centuries. Similarly, the uncertainty about the true volatility of growth magnifies long-run risks, as does the ambiguity surrounding the probability of rare macroeconomic catastrophes. Our aim in this paper is to provide a systematic analysis of the impact of parametric uncertainty on asset prices in the standard consumption-based

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CAPM. It is shown that the parameter uncertainty that affects the random walk of the consumption growth rate has no effect on the price of short-term bond and consumption claims. [1] For longer maturities, our benchmark is a model in which the term structures are known to be flat in the absence of parameter uncertainty. Under the discounted expected utility model with constant relative risk aversion, this requires the growth of log consumption to be governed by a random walk – meaning that increments are stationary and serially independent. Under this assumption, the growth process is characterized by the distribution of increments in log consumption. This distribution is subject to some parameter uncertainty.

Our main results are generic and hold without making any restriction on this distribution or on the nature of the parameter uncertainty. It can include ambiguity related to the trend of growth, the volatility of growth, or the frequency of macro catastrophes for example. The mathematical methodology used in this paper is based on the Cumulant-Generating Function (CGF) associated to the distribution of log consumption. Martin (2013) recently used the properties of CGF to characterize asset prices when the growth of log consumption is not Gaussian. This paper provides various illustrations of the power of this method which we improve in the context of parameter uncertainty. Under our assumptions, the risk-free rate and the risk premium can be obtained by performing sequences of two CGF calculations, the first on the conditional increment of log consumption, and the second on the conjugate distribution of the uncertain parameters.

As is well-known (see e.g. Billingsley, 1995), a probability distribution can be represented by its vector of cumulants, so that the parameter uncertainty affecting the distribution of increments can be characterized by the joint distribution of its cumulants. It is shown that parameter uncertainty has *no* effect on short-term risk-adjusted discount rates. To be more precise, the efficient instantaneous risk-adjusted discount rates are obtained by applying the standard CAPM pricing formulas using the expected cumulants as if they would be the true values. In words, this implies that mean-preserving spreads in the distribution of cumulants have no impact on the price of very short-term zero-coupon bonds and equity in our framework. This generalizes a result by Veronesi (2000) who has demonstrated that the short-term risk-free interest rate is not affected by parameter uncertainty when this uncertainty affects the drift rate of aggregate consumption, i.e. when only the first cumulants is uncertain.[2] This is due to the fact that the uncertainty affecting the expected growth has no impact on the conditional posterior volatility of consumption in the short run. This result is also reminiscent of a result by Hansen and Sargent (2010) who showed that parameter uncertainty does not contribute to local uncertainty prices in a Bayesian analysis.

Gollier (2008, 2012) shows that the term structure of interest rates is decreasing in the standard consumption-based CAPM with a prudent representative agent whenever the growth rates of consumption are positively serially correlated, i.e., when shocks to the growth rate are persistent. This is because the persistence of shocks tends to magnify the uncertainty affecting long-run consumption, thereby inducing the prudent representative agent to favor investments yielding sure benefits for the distant future. This reduces the long-term interest rate. Our generic model with parametric uncertainty is a special case since, under parametric uncertainty, the observation of a large growth rate in the short run yields a permanent optimistic revision of beliefs. As observed by Collin-Dufresne al. (2016), posterior moments of growth are martingales under rational expectations, so that shocks to beliefs about the distribution of increments in log consumption are permanent. This magnifies the long-run consumption risk.

Adjusting the discount rates to risk requires estimating the maturity-specific risk premia. Because the persistence of learning shocks magnifies the long-run aggregate risk compared to this benchmark case, it raises the aggregate long-term risk premium. This provides an intuition to our result that the term structure of the aggregate risk premia is increasing. This property holds if the set of uncertain cumulants are independently distributed. However, this property does not hold in general. This is because the risk premium is also affected by the higher cumulants of log consumption outside the Gaussian world. For example, if the trend and the volatility of growth are negatively correlated, then the term structure of the annualized skewness of log consumption is locally decreasing. This decreasing skewness effect can be stronger than the increasing variance effect to make the term structure of risk premia decreasing.

The persistence of shocks to beliefs has an ambiguous effect on the long-term risk-adjusted discount rate because it reduces the risk-free rate and it potentially raises the risk premium. If the asset's beta is large enough, the net effect may be positive, yielding an increasing term structure of the risk-adjusted discount rates. If the parameter uncertainty affects cumulants in a statistically independent way, we show that the risk-adjusted discount rate is reduced by the uncertainty if and only if the consumption CAPM beta of the asset is smaller than half the degree of relative risk aversion. This suggests that parameter uncertainty should induce us to invest more for the distant future if the investment opportunity set contains enough projects with a small beta.

These generic results are presented in Section 3. Following the current trend of the literature, we explore in Section 4 and 5 some special cases to quantify these effects. In Section 4, it is assumed that the economy may face macroeconomic catastrophes at low frequency. In normal time, the growth of log consumption is Gaussian, but a large drop in aggregate consumption strikes the economy at infrequent dates. Our modeling duplicates the one proposed by Barro (2006, 2009), except for the recognition of the existence of some parametric uncertainty. Martin (2013) convincingly demonstrates that it is complex to estimate the true probability of infrequent catastrophes, and that a small modification in the choice of

[1] Brennan (1997) was the first to examine the term structure of risk premia. Ang and Liu (2004) refer to the notion of "spot discount rates", which is parallel to the "prices of zero-coupon equity" and "dividend strips" in Lettau and Wachter (2007).

[2] Veronesi (2000) considers a Markov multiple-regime switch process for the growth trend. We refer here to the special case in which the growth trend is permanent.

parameters values has a huge effect on asset prices. This remark is taken into account by explicitly introducing ambiguity about this probability into the model. As explained earlier, this addition does not affect the instantaneous risk-free rate and the instantaneous aggregate risk premium obtained by Barro. But it dramatically affects the term structures of risk-adjusted discount rates. In a calibration exercise based on Barro (2009) and using a Bayesian approach to calibrate our degree of ignorance on the probability of catastrophes, it is shown that the risk-free rate and the risk premium for long maturities are strongly impacted by the uncertainty affecting the probability of catastrophes.

Section 5 is devoted to a model in which log consumption follows an arithmetic Brownian motion whose trend or volatility is uncertainty. Weitzman (2007) showed that the standard asset pricing puzzles can easily be reversed once it is recognized that the volatility of the growth of aggregate consumption is uncertain. Weitzman calibrated this uncertainty in such a way that risk-adjusted discount rates are unbounded at any maturity. Our aim in that framework is also to characterize the term structures for bonds and equity without restricting the model to some specific beliefs about the trend of growth. This analysis is generalized in Section 6 to the case of persistent shocks taking the form of an autoregressive structure for the growth rate of consumption.

These applications illustrate our main finding that parametric uncertainty tends to make the term structure of systematic risk premia increasing. This is puzzling, because a growing empirical literature demonstrates that actual equity premia have a *decreasing* term structure. This fact has long been supported by the observation that value stocks have a higher average return than growth stocks, i.e., stocks with a larger expected growth of dividends. More recently, van Binsbergen et al. (2012, 2013) and van Binsbergen and Koijen (2016) have shown that the term structure of equity is decreasing for maturities up to 10 years, using first option prices in the United States, and then dividend futures on the S&P500, the Eurostoxx 50, the FTSE 100, and the Nikkei 225 index. Giglio et al. (2015) and Giglio et al. (2015) show that the term structure of housing discount rates is decreasing for maturities up to 100 years. In Section 7, we review some plausible resolutions of this puzzle. As claimed by Croce et al. (2015), one of them is that the representative agent cannot observe the persistent component of growth. This introduces a new source of parametric uncertainty. But because the role of the current value of the persistent component for the determination of long-run growth vanishes for long maturities, this parametric uncertainty raises systematic risk premia only for intermediate maturities, thereby making the term structure of risk premia hump-shaped.

## 2. The model

A marginal investment project whose cash flow of net benefits (or dividend) is represented by a continuous-time stochastic process $\{D_t : t \geq 0\}$ is evaluated at date 0. In order to value this project, its impact on the intertemporal social welfare

$$W_0 = E\left[\int_0 e^{-\delta t} u(c_t) dt | I_0\right] \tag{1}$$

is measured, where $u$ is the increasing and concave utility function of the representative agent, $\delta$ is her rate of pure preference of the present, $\{c_t | t \geq 0\}$ is the continuous-time process of consumption of the representative agent, and $I_0$ is the information set available at $t=0$. We assume that this integral exists. Because we limit the analysis to the pricing of the asset at date 0, we hereafter make the contingency of the expectation operator to $I_0$ implicit. Because the investment project is marginal, its implementation increases intertemporal social welfare if and only if

$$\int_0 E\left[\frac{e^{-\delta t} D_t u'(c_t)}{u'(c_0)}\right] dt \geq 0. \tag{2}$$

We assume that this integral is bounded. This inequality can be rewritten as a standard NPV formula:

$$P_0 = \int_0 e^{-\rho_t t} ED_t dt \geq 0, \tag{3}$$

where $\rho_t$ is the rate at which the expected cash flow occurring in $t$ years should be discounted. In a frictionless economy, the NPV $P_0$ would be the equilibrium price at date 0 of the asset that would deliver the cash flow $\{D_t : t \geq 0\}$. It is important to observe at this stage that the asset is decomposed into a bundle of horizon-specific dividends, and that a specific discount rate $\rho_t$ is used for each of them. Thus, $\rho_t$ can also be interpreted as the expected yield-to-maturity of the zero-coupon claim $D_t$ with maturity $t$. It is characterized by the following equation:

$$\rho_t = \delta - \frac{1}{t} \ln \frac{ED_t u'(c_t)}{u'(c_0) ED_t} = r_{ft} + \pi_t. \tag{4}$$

It is traditional in the consumption CAPM to decompose the project-specific discount rate $\rho_t$ into a risk-free discount rate $r_{ft}$ and a project-specific risk premium $\pi_t$. From (4), we define these two components of the discount rate as follows:

$$r_{ft} = \delta - \frac{1}{t} \ln \frac{Eu'(c_t)}{u'(c_0)}, \tag{5}$$

$$\pi_t = -\frac{1}{t} \ln \frac{ED_t u'(c_t)}{ED_t Eu'(c_t)}. \tag{6}$$

Whereas the risk-free rate $r_{ft}$ only depends upon the distribution of $c_t$, the estimation of the project-specific risk premium $\pi_t$ requires knowing the joint probability distribution of $(c_t, D_t)$. Because the risk premium $\pi_t$ is zero when the project is safe and more generally when its future cash flow is independent of future aggregate consumption, $r_{ft}$ is indeed the rate at which safe projects should be discounted. Throughout the paper, we assume that $u'(c)=c^{-\gamma}$and that for each date $t$, there exists $\beta_t \in \mathbb{R}$ so that

$$E[D_t|c_t]=c_t^{\beta}. \tag{7}$$

$\beta$ is the elasticity of the net benefit to changes in aggregate consumption. Under some conditions, it is also the consumption CAPM beta of the net benefit at date $t$.[3] In the Gaussian world, this implies that the expected growth of log dividend is linear in the growth of log consumption, as is the case in most models in consumption-based asset pricing theory.

We characterize the properties of asset prices by using a standard tool in statistics. Let $\chi(a,x)=\ln E\exp(ax)$ denote the Cumulant-Generating Function (CGF) associated to random variable $x$ evaluated at $a\in\mathbb{R}$. The CGF function, if it exists, is the log of the better known moment-generating function. Following Martin (2013), asset pricing formulas (5) and (6) can now be rewritten using CGF functions:

$$r_{ft}=\delta-t^{-1}\chi(-\gamma, G_t), \tag{8}$$

and $\pi_t=\pi_t(\beta_t)$ with

$$\pi_t(\beta)=t^{-1}(\chi(\beta, G_t)-\chi(\beta-\gamma, G_t)+\chi(-\gamma, G_t)), \tag{9}$$

where $G_t=\ln c_t/c_0$ is log consumption growth. Notice that, ignoring $\delta$, $\chi(\beta, G_t)$, $\chi(\beta-\gamma, G_t)$ and $-\chi(-\gamma, G_t)$ can be interpreted respectively as the log of expected payoff, the log of price, and the log of the risk-free return. This means that the right side of Eq. (9) is the different between the expected annualized return $t^{-1}(\chi(\beta, G_t)-\chi(\beta-\gamma, G_t))$ of $D_t$ over the risk-free rate. Until recently, the modern theory of finance used to focus the analysis mostly on the equity premium, i.e., on the risk premium associated to a perpetual claim on aggregate consumption. Following van Binsbergen et al. (2012), we are interested in characterizing the full term structure of risk premia $\pi_t(\beta)$ in parallel to the term structure of risk-free rates $r_{ft}$.

In this paper, we use the following properties of CGF (see Billingsley (1995)).

**Lemma 1.** : *If it exists, the CGF function $\chi(a,x)=\ln E\exp(ax)$ has the following properties*:

i. $\chi(a,x)=\sum_{n=1}^{\infty}\kappa_n^x a^n/n!$ *where $\kappa_n^x$ is the nth cumulant of random variable x. If $m_n^x$ denotes the centered moment of x, we have that* $\kappa_1^x=Ex$, $\kappa_2^x=m_2^x$, $\kappa_3^x=m_3^x$, $\kappa_4^x=m_4^x-3(m_2^x)^2, \ldots$.
ii. *The most well-known special case is when x is $N(\mu,\sigma^2)$, so that $\chi(a,x)=a\mu+0.5a^2\sigma^2$.*
iii. $\chi(a,x+y)=\chi(a,x)+\chi(a,y)$ *when x and y are independent random variables.*
iv. $\chi(0,x)=0$ *and $\chi(a,x)$ is infinitely differentiable and convex in a.*
v. $a^{-1}\chi(a,x)$ *is increasing in a, from Ex to the supremum of the support of x when a goes from zero to infinity.*

Property *i* explains why $\chi$ is called the cumulant-generating function, and it links the sequence of cumulants to those of the centered moments. The first cumulant is the mean. The second, the third and the fourth cumulants are respectively the variance, the skewness and the excess kurtosis of the random variable. Because the cumulants of the normal distribution are all zero for orders $n$ larger than 2, the CGF of a normally distributed $x$ is a quadratic function of $a$, as expressed by property *ii*. This property also implies that the CGF of a Dirac distribution degenerated at $x=x_0\in\mathbb{R}$ is equal to $ax_0$. Property $v$ will play a crucial role in this paper. It is a consequence of property *iv*, which is itself an illustration of the Cauchy-Schwarz inequality.

In the asset pricing literature, the simplest configuration to calibrate Eqs. (8) and (9) is when $G_t$ is normally distributed, so that we can use property *ii* of Lemma 1 to rewrite Eq. (9) as follows:

$$\pi_t(\beta)=\beta\gamma v_t \tag{10}$$

where $v_t=t^{-1}Var(\log c_t)$ is the annualized variance of log consumption. In this specific Gaussian specification, the term structure of risk premia is proportional to the term structure of the annualized variance of log consumption contingent to current information. Hansen et al. (2008) pursued the same goal of describing the term structure of risk prices in a Gaussian world with persistent shocks to growth.

In this paper, we apply this generic consumption pricing model to the case of parameter uncertainty. Namely, we assume that the stochastic process governing the evolution of $G_t$ is a function of an unknown parameter $\theta\in\Theta$. The current beliefs about $\theta$ are represented by some distribution function with support in $\Theta$.[4] Parameter uncertainty usually implies that the unconditional distribution of log consumption is not normal. This is an important source of complexity because, contrary to the Gaussian case examined in the previous section, Eq. (9) will entail cumulants of order larger than 2.

[3] See also Campbell (1986) and Martin (2013). Weitzman (2013) considers an alternative risk profile $D_t=a+b_tc_t$which can be interpreted as a portfolio containing $a$ units of the risk-free asset and an exponentially-decreasing equity share $b_t$.

[4] The observation of future changes in consumption will allow for a Bayesian updates of beliefs which will impact prices. In this framework with exponential Discount Expected Utility, agents' decisions (and so asset prices) will be time consistent. In this paper, we are interested in characterizing asset prices *today*. We do not examine the dynamic of prices, so we do not determine how beliefs are updated. We just do not need to do this for our purpose.

Let us discretize time by periods of equal duration $\Delta > 0$, which can be arbitrary small. We hereafter crucially assume that, conditional on any $\theta \in \Theta$, $G_t = \ln c_t/c_0$ is a stationary random walk. This means that, conditional on $\theta$, temporal increments in $G_t$ are serially independent and stationary.[5] In this paper, we examine the properties of asset prices under this general assumption. This is a clear difference with respect to recent papers in mathematical finance (see e.g. Hansen et al., 2008; Hansen and Scheinkman, 2009; Hansen, 2012) in which the focus is on the effect of unambiguous auto-regressive components of the growth process on the long-term risk prices. Our findings rely on the following theorem.

**Theorem 1.** : *Suppose that, conditional on any $\theta \in \Theta$, log consumption is a random walk whose increment is distributed as $g|\theta$. Then, for all $t > 0$:*

$$r_{ft} = \delta - t^{-1}\chi(t, \chi(-\gamma, g|\theta)). \tag{11}$$

and for all $\beta \in \mathbb{R}$:

$$\pi_t(\beta) = t^{-1}(\chi(t, \chi(\beta, g|\theta)) - \chi(t, \chi(\beta-\gamma, g|\theta)) + \chi(t, \chi(-\gamma, g|\theta))). \tag{12}$$

**Proof:** Let $g_{t,\Delta} = G_{t+\Delta} - G_t = \ln c_{t+\Delta}/c_t$ denote the growth of log consumption during the interval of time $[t, t+\Delta]$. We can then rewrite Eq. (8) as follows:

$$\begin{aligned} r_{ft} &= \delta - t^{-1} \ln E\left[E\left[e^{-\gamma \sum_{i=1}^{t/\Delta} g_{(i-1)\Delta,\Delta}}|\theta\right]\right] \\ &= \delta - t^{-1} \ln E\left[E\left[e^{-\gamma g}|\theta\right]^{t/\Delta}\right] \\ &= \delta - t^{-1} \ln E e^{t\chi(-\gamma, g|\theta)/\Delta} \\ &= \delta - t^{-1}\chi\left(t, \chi\left(-\gamma, g|\theta\right)/\Delta\right), \end{aligned}$$

with $g_\Delta = \ln c/c_0$. The second equality above is a consequence of the assumption that $G_t|\theta$ is a random walk, so that the $g_{j\Delta,\Delta}$ are i.i.d. variables. The last two equalities are direct consequences of the definition of the CGF. A similar exercise can be performed on Eq. (9) for the risk premium, which implies Eq. (12). □

Eq. (11) means that the risk-free rate is determined by a sequence of two CGF operations. One must first compute $x|\theta = \chi(-\gamma, g|\theta)$, which is the CGF of consumption growth $g$ conditional on $\theta$. One must then compute $\chi(t, x)$ by using the distribution of $\theta$ that characterizes our current beliefs about this unknown parameter. The comparison of Eqs. (8) and (11) indicates the very specific structure that conditional random walk and parameter uncertainty bring to the asset pricing model.

In the benchmark case where log consumption follows an arithmetic Brownian motion with trend $\mu$ and volatility $\sigma$, $G_t$ is normally distributed with mean $\mu t$ and variance $\sigma^2 t$. Using property *ii* in Lemma 1 implies that Eqs. (11) and (12) can be rewritten as follows

$$r_{ft} = \delta + \gamma\mu - 0.5\gamma^2\sigma^2, \tag{13}$$

and

$$\pi_t(\beta) = \beta\gamma\sigma^2. \tag{14}$$

Eq. (13), which is often referred to as the extended Ramsey rule, holds independent of the maturity of the cash flow. In other words, the term structure of the safe discount rate is flat under this benchmark specification. As is well-known, the last term in the right-hand side of Eq. (13) measures a precautionary effect. It comes from the observation that consumers want to invest more for the future when this future is more uncertain (Drèze and Modigliani, 1972; Kimball, 1990). This tends to reduce the discount rate. The precautionary effect is proportional to the volatility of the growth of log consumption. Eq. (14) tells us that the project-specific risk premium $\pi_t(\beta)$ is just equal to the product of the project-specific beta by the aggregate risk premium $\pi = \gamma\sigma^2$. Compared to historical market returns, the standard calibration of these two equations yields a too large risk-free rate (risk-free rate puzzle, Weil (1989)) and a too small risk premium (equity premium puzzle, Grossman and Shiller, 1981; Hansen and Singleton, 1983; Mehra and Prescott, 1985). Notice that the risk-adjusted discount rate $\rho_t(\beta) = r_{ft} + \pi_t(\beta)$ is increasing in the aggregate uncertainty measured by $\sigma^2$ if $\beta$ is larger than $\gamma/2$, in which case the risk premium effect dominates the precautionary effect.

## 3. Generic properties of the term structures of risk-adjusted discount rates

In this section, we derive the generic properties of $r_{ft}$ and $\pi_t(\beta)$ as defined in Theorem 1. We first examine the role of parameter uncertainty on instantaneous risk-adjusted discount rates. This requires us to go to the limit case of continuous-time ($t = \Delta \to 0$). Using property $v$ of Lemma 1 we immediately obtain that

$$r_{f0} = \delta - E\chi(-\gamma, g|\theta)$$

[5] If we require this assumption to hold independent of the duration $\Delta$, log consumption must be governed by a Lévy process.

$$= \delta - E\left[\sum_{n=1}^{\infty} \frac{(-\gamma)^n}{n!}\kappa_n^{g|\theta}\right] = \delta + \sum_{n=1}^{\infty} (-1)^{n-1}\frac{\gamma^n}{n!}E\left[\kappa_n^{g|\theta}\right]. \tag{15}$$

This tells us that the instantaneous discount rate is not affected by parametric uncertainty in the sense that only the *expected* cumulants of the change in log consumption matter to compute it. From Eq. (12), the same result holds for risk premia, with

$$\begin{aligned}\pi_0 &= E\chi(\beta, g|\theta) - E\chi(\beta-\gamma, g|\theta) + E\chi(-\gamma, g|\theta) \\ &= \sum_{n=2}^{\infty} \frac{\beta^n - (\beta-\gamma)^n + (-\gamma)^n}{n!}E\left[\kappa_n^{g|\theta}\right].\end{aligned} \tag{16}$$

Only the expected variance, skewness, kurtosis and higher cumulants matter to compute the instantaneous risk premium when these cumulants are uncertain.

**Proposition 1.** *Suppose that log consumption conditional on $\theta$ follows a random walk, for all $\theta \in \Theta$. This parametric uncertainty has no effect on the instantaneous risk-free rate and risk premium in the sense that they only depend upon the expected cumulants of the incremental log consumption.*

This generalizes earlier observations that parameter uncertainty does not affect short-lived asset prices (Veronesi, 2000; Hansen and Sargent, 2010; Collin-Dufresne et al., 2016). The intuition of this result can be derived from the observation that the annualized variance of log consumption satisfies the law of total variance:

$$t^{-1}Var\ \ln c_t = EVar(g|\theta) + tE\left[\left(E[g|\theta] - E[g]\right)^2\right]. \tag{17}$$

Because the second term in the RHS of this equation vanishes when $t$ tends to zero, it tells us that, for short maturities, the annualized variance of log consumption is unaffected by a mean-preserving transformation of the probability distribution of increments.[6] This result illustrates the fact that when valuing a short-term claim, investors care about the short-term dividend growth, but they do not care about news concerning more distant consumption growth. This is specific to the Discounted Expected Utility à la Lucas (1978). Collin-Dufresne et al. (2016) show that Proposition 1 is not robust to the introduction of Epstein-Zin preferences precisely because it makes the representative agent caring about long-run risks when evaluation instantaneous cash flows.

But parameter uncertainty does affect the discount rate associated to longer maturities. Indeed, by application of property $v$ of Lemma 1, the risk-free discount rate has a decreasing term structure when the parametric uncertainty is such that $\chi(-\gamma, g|\theta)$ is sensitive to $\theta$. The intuition of this result can be derived from Eq. (17) which shows that the annualized variance of log consumption has an increasing term structure when the trend of growth is uncertain. This observation generalizes a result obtained by Gollier (2008) in the special case of a geometric Brownian motion for aggregate consumption with an uncertain trend.

**Proposition 2.** Suppose that log consumption conditional on $\theta$ follows a random walk, for all $\theta \in \Theta$. This parametric uncertainty makes the term structure of risk-free discount rates monotonically decreasing.

One can estimate the slope of this term structure for small maturities by observing from property 1 of Lemma 1 that the derivative of $t^{-1}\chi(t,x)$ at $t=0$ equals $0.5Var(x)$. Applying this to Eq. (11) implies that

$$\left.\frac{\partial r_{ft}}{\partial t}\right|_{t=0} = -0.5Var(\chi(-\gamma, g|\theta)) = -0.5Var\left(\sum_{n=1}^{\infty} \frac{(-\gamma)^n}{n!}\kappa_n^{g|\theta}\right). \tag{18}$$

In the special case in which uncertain cumulants are not correlated, this equation simplifies to

$$\left.\frac{\partial r_{ft}}{\partial t}\right|_{t=0} = -\sum_{n=1} \frac{\gamma^{2n}}{2(n!)^2}Var\left(\kappa_n^{g|\theta}\right). \tag{19}$$

We see that the slope of the term structure is increasing in absolute value in the degree of uncertainty measured by the variance of the unknown cumulants.

The analysis of the term structure of aggregate risk premia is more complex because the risk premium is the sum and difference of three different double-CGF functionals. Observe that property $i$ of Lemma 1 applied to Eq. (12) implies that

$$\left.\frac{\partial \pi_t(\beta)}{\partial t}\right|_{t=0} = 0.5(Var(\chi(\beta, g|\theta)) + Var(\chi(-\gamma, g|\theta)) - Var(\chi(\beta-\gamma, g|\theta))). \tag{20}$$

In the special case in which the uncertain cumulants are statistically independent, we obtain that

$$\left.\frac{\partial \pi_t(\beta)}{\partial t}\right|_{t=0} = \sum_{n=1} \frac{\beta^{2n} + \gamma^{2n} - (\beta-\gamma)^{2n}}{2(n!)^2}Var\left(\kappa_n^{g|\theta}\right). \tag{21}$$

[6] This observation is not a proof of Proposition 1 since the other cumulants of log consumption should also matter for the determination of the risk-free rate.

In the proof of the following proposition, we show that the coefficients of $Var\left(\kappa_n^{g|\theta}\right)$ have the same sign than $\beta$, which implies that the term structure of the aggregate risk premia is locally increasing.

**Proposition 3.** *Suppose that log consumption conditional on $\theta$ follows a random walk, for all $\theta \in \Theta$. Suppose also that the uncertain cumulants of its increments are statistically independent. This implies that the risk premium $\pi_t(\beta)$ has an increasing (resp. decreasing) term structure for small maturities when $\beta > 0$ (resp. $\beta < 0$).*

**Proof:** From Eq. (21), we would be done if $f_\beta(\gamma) = \beta^{2n} + \gamma^{2n} - (\beta-\gamma)^{2n}$ has the same sign as $\beta$. Observe that

$$f'_\beta(\gamma) = 2n\left(\gamma^{2n-1} + (\beta-\gamma)^{2n-1}\right). \tag{22}$$

This equation implies that $f'_\beta(0)$ has the same sign as $\beta$. Because $f'_\beta$ never alternate in sign, we can conclude that $f_\beta$ is monotonically increasing (resp. decreasing) in $\gamma$ when $\beta$ is positive (resp. negative). Because $f_\beta(0) = 0$, we obtain that $f_\beta(\gamma)$ has the same sign as $\beta$ for all $\gamma \in \mathbb{R}^+$. □

However, contrary to what we obtained in Proposition 2 for the risk-free rate, Proposition 3 is not robust to the introduction of correlation among two or more cumulants. To show this, suppose that cumulants of degrees $m$ and $n$ are the only two uncertain cumulants of g. In that case, it is easy to show from Eq. (20) that

$$\left.\frac{\partial \pi_t(\beta)}{\partial t}\right|_{t=0} = \frac{\beta^{2m} + \gamma^{2m} - (\beta-\gamma)^{2m}}{2(m!)^2} Var\left(\kappa_m^{g|\theta}\right) + \frac{\beta^{2n} + \gamma^{2n} - (\beta-\gamma)^{2n}}{2(n!)^2} Var\left(\kappa_n^{g|\theta}\right) + \frac{\beta^{m+n} + (-\gamma)^{m+n} - (\beta-\gamma)^{m+n}}{m!n!} Cov\left(\kappa_m^{g|\theta}, \kappa_n^{g|\theta}\right). \tag{23}$$

This equation is useful to show that the term structure of risk premia is not necessarily increasing. Indeed, a growing literature documents evidence that the term structure of the equity premium is downward sloping for time horizons standard for financial markets.[7] Observe however that Eq. (23) is compatible with a downward sloping term structure for $\pi_t(1)$ if the covariance term is sufficiently negative, since the two variance terms are positive. Such an example is developed in Appendix. This example illustrates the fact that parameter uncertainty does not only make the annualized variance of log consumption increasing. It also affects the term structures of its annualized skewness and excess kurtosis. Although that cannot reverse the result that the term structure of risk-free rates is decreasing, it can make the term structure of risk premia decreasing when the uncertain cumulants of log consumption are statistically dependent.

Proposition 3 makes it unclear whether parameter uncertainty actually raises the global willingness to invest for the medium and distant future. If the betas of investment projects are large enough, the parameter uncertainty should reduce the intensity of investments, because it will raise the risk premium more than it will reduce the risk-free rate. It is interesting to determine the critical beta at which the risk-adjusted discount rate $\rho_t(\beta) = r_{ft} + \pi_t(\beta)$ has a flat term structure in the neighborhood of $t=1$. Suppose again that the uncertain cumulants are statistically independent. From Eqs. (19) and (21), the risk-adjusted discount rate is such that

$$\left.\frac{\partial \rho_t(\beta)}{\partial t}\right|_{t=0} = \sum\nolimits_{n=1} \frac{\beta^{2n} - (\beta-\gamma)^{2n}}{2(n!)^2} Var\left(\kappa_n^{g|\theta}\right) \tag{24}$$

This implies that $\rho_t$ is locally increasing in $t$ if and only if $\beta^2$ is larger than $(\beta-\gamma)^2$, i.e., if and only if $\beta$ is larger than $\gamma/2$.

**Corollary 1.** *Under the assumptions of Proposition 3, the term structure of the risk-adjusted discount rate $\rho_t(\beta)$ is increasing (resp. decreasing) for small maturities if $\beta$ is larger (resp. smaller) than $\gamma/2$.*

There is a simple intuition for this result. It combines the observation that the parametric uncertainty magnifies the uncertainty affecting the future level of development of the economy with the observation made at the end of the previous section that long-run risk decreases or increases the discount rate depending upon whether $\beta$ is smaller or larger than $\gamma/2$. Because it is often assumed in the calibration of macro-financial models that relative risk aversion is much larger than 2, we anticipate that most assets should have a beta smaller than $\gamma/2$. Thus, for a number of assets, we should observe a decreasing term structure of discount rates. This is compatible with the observation made by Giglio et al. (2015) that the term structure of housing discount rates is decreasing.

Corollary 1 implies that the long discount rate of an asset with risk profile $\beta = 1$ is increased by the uncertainty affecting the cumulants of log consumption if and only if relative risk aversion is smaller than 2. Barsky (1989) and Abel (2002) examined the impact on asset prices of a small mean-preserving increase in risk in future consumption. They showed that it increases the risk-adjusted discount rate of consumption claims ($\beta = 1$) if and only if relative risk aversion is less than 1. The discrepancy between the two results originates from the fact that these authors assume a constant expected consumption, whereas we assume a constant expected log consumption. Our ceteris paribus assumption implies an increase in expected consumption. It generates an additional increase in all discount rates. This is why our condition is less demanding.

[7] See for example van Binsbergen et al. (2012) and the references mentioned in that paper.

The result described in Corollary 1 is not robust to the introduction of correlation among the uncertain cumulants. If the cumulants of degrees $m$ and $n$ are the only two uncertain cumulants of $g$, we obtain that

$$\left.\frac{\partial \rho_t(\beta)}{\partial t}\right|_{t=0} = \frac{\beta^{2m}-(\beta-\gamma)^{2m}}{2(m!)^2}Var\left(\kappa_m^{g|\theta}\right)+\frac{\beta^{2n}-(\beta-\gamma)^{2n}}{2(n!)^2}Var\left(\kappa_n^{g|\theta}\right) + \frac{\beta^{m+n}-(\beta-\gamma)^{m+n}}{m!n!}Cov\left(\kappa_m^{g|\theta},\kappa_n^{g|\theta}\right). \tag{25}$$

In the special case of $\beta=\gamma/2$, the first two terms in the RHS of this equation are zero. But the coefficient of the covariance in the third term is clearly positive if $m+n$ is an odd integer. This implies that the term structure of the risk-adjusted discount rate with $\beta=\gamma/2$ will not be flat when the uncertain cumulants are correlated. In particular, a positive correlation between two subsequent cumulants tends to make this term structure of $\rho_t(\beta=\gamma/2)$ increasing, at least at small maturities.

The methodology based on the CGF proposed by Martin (2013) is also useful to explore the curvature of the term structures. From Lemma 1, we have that

$$\left.\frac{\partial^2 \rho_t(\beta)}{\partial t^2}\right|_{t=0} = \frac{1}{3}Skew(\chi(\beta,g|\theta)-\chi(\beta-\gamma,g|\theta)), \tag{26}$$

where $Skew(x)$ is the skewness of $x$. If the uncertainty on the distribution of log consumption is concentrated on the $n$th cumulant, then this equation simplifies to

$$\left.\frac{\partial^2 \rho_t(\beta)}{\partial t^2}\right|_{t=0} = \frac{\beta^{3n}-(\beta-\gamma)^{3n}}{3(n!)^3}Skew\left(\kappa_n^{g|\theta}\right). \tag{27}$$

This means for example that the term structure of the discount rate for the aggregate risk ($\beta=1$) is convex at $t$=0 if the trend ($n$=1) of growth is uncertain and positively skewed. Pursuing in the same vein for larger derivatives of $\rho_t$ would allow us to fully describe the shape of the term structure of discount rates from the sign of the successive cumulants of $g$.

In the spirit of Hansen and Scheinkman (2009) and Hansen (2012), we now determine the asymptotic values of the discount rates in the context of parameter uncertainty. Property $v$ of Lemma 1 immediately yields the following result. The suprema are taken with respect to $\theta \in \Theta$.

**Proposition 4.** *Suppose that log consumption conditional on $\theta$ follows a random walk, for all $\theta \in \Theta$. Assuming that the support of $\chi(k,g|\theta)$ has an upper bound for $k \in \{\beta,-\gamma,\beta-\gamma\}$, this implies that*

$$\lim_{t\to\infty} r_{ft} = \delta - \sup\chi(-\gamma,g|\theta). \tag{28}$$

$$\lim_{t\to\infty}\pi_t(\beta) = \sup\chi(\beta,g|\theta)+\sup\chi(-\gamma,g|\theta)-\sup\chi(\beta-\gamma,g|\theta) \tag{29}$$

$$\lim_{t\to\infty}\rho_t(\beta) = \delta+\sup\chi(\beta,g|\theta)-\sup\chi(\beta-\gamma,g|\theta). \tag{30}$$

Eq. (28) tells us that the asymptotic risk-free rate corresponds to the smallest possible conditional risk-free rate. This is reminiscent of similar results obtained by Weitzman (1998, 2001) in another context. Notice that we can infer from Eq. (29) that the symmetric result for the risk premium does not hold, because the sum of suprema is generally not equal to the supremum of the sum. Thus, it is not true in general that the asymptotic risk premium equals the largest possible conditional risk premium. A counterexample is presented in the next section.

Propositions 1 and 2 imply that parametric uncertainty reduces interest rates. This implies in turn that it raises the price of all safe assets, except the instantaneous zero-coupon bond. In fact, parametric uncertainty has a non-negative effect on the price of all assets in the economy, as we show now. Consider first an asset delivering a single claim $c_t^{\beta_t}$ in $t$ periods. Its price today equals

$$P = e^{-\delta t}\frac{Ec_t^{\beta_t}u'(c_t)}{u'(c_0)} = c_0^{\beta_t}e^{-\delta t}Ee^{(\beta_t-\gamma)G_t} = c_0^{\beta_t}e^{-\delta t}E\left[e^{t\chi(\beta_t-\gamma,g|\theta)}\right]. \tag{31}$$

The expectation in the right side of the last equality is taken with respect to $\theta$. Then, by the Jensen inequality, we have that for all $t>0$,

$$P \geq c_0^{\beta_t}e^{-\delta t}e^{tE\chi(\beta_t-\gamma,g|\theta)} = \overline{P}. \tag{32}$$

$\overline{P}$ is the asset price that would prevail at equilibrium if all uncertain cumulants would be replaced by their expectation. Thus, we can conclude from Eq. (32) that parametric uncertainty raises this asset price. This is true for all maturities $t$ and for all risk profiles $\beta_t$. Because any asset is a portfolio of future claims, this result demonstrates the following proposition. An asset is "long" if its flow of dividends $(D_1,D_2,\ldots)$ are such that, for all $t$, there exists a pair $(\beta_t,h_t)$of scalars with $h_t \geq 0$ so that $D_t = h_t c_t^{\beta_t}$almost surely.

**Proposition 5.** *Suppose that log consumption conditional on $\theta$ follows a random walk, for all $\theta \in \Theta$. Then, the price of any long asset is weakly increasing in the parametric uncertainty affecting the cumulants of log consumption.*

More generally, any mean-preserving spread in the distribution of a cumulant of log consumption raises the price of all assets in the economy. The limit case is when $\beta$ equals $\gamma$, which implies that the marginal utility of future dividends is maturity-independent. Therefore, parameter uncertainty has no effect on the price of these assets with $\beta=\gamma$, as easily observed from Eq. (31).

It may appear surprising that all assets prices are weakly increasing with the uncertainty affecting the growth process, since we have seen earlier that if $\beta$ is large enough, then the risk-adjusted discount rate is increasing in the degree of uncertainty. This tends to reduce the price of the riskiest assets. But at the same time, parametric uncertainty tends to raise expected dividends. Indeed, we have that

$$Ec_t^{\beta_t}=c_0^{\beta_t}Ee^{\beta_t G_t}=c_0^{\beta_t}E\left[e^{t\chi(\beta_t,g|\theta)}\right]. \tag{33}$$

By Jensen inequality, this expected dividend is also increased by parametric uncertainty. This is because of the intertemporal compounding of growth rates. Proposition 5 tells us that this expected dividend effect always dominates the discounting effect, so that any long asset price is increasing in the degree of uncertainty prevailing in the economy.

Proposition 5 tells us that if exogenous shocks to the degree of uncertainty affecting the cumulants of growth are the main determinant of the fluctuation of assets prices, then these prices should comove in the same direction. But since the pioneering works by Shiller (1982), we know that there is little comovements between the price of risky assets, bonds and land. But suppose alternatively that news affect the beliefs of the representative agent in such a way that they generate first-order stochastic dominant shifts in the distributions of $g|\theta$, for all $\theta$. This may be due to the observation of a large growth rate of consumption. From Eq. (31), such shocks raise the price of all assets whose beta is larger than $\gamma$, and they reduce the price of all other assets. The prices of bonds and of a number of risky assets would move in opposite direction in that case.

## 4. Application 1: Unknown probability of a macroeconomic catastrophe

In the spirit of Barro (2006, 2009), Backus et al. (2011) and Martin (2013), we now examine a discrete version of a mixture of Brownian and Poisson processes. Rare events have recently been recognized for being a crucial determinant of assets prices. The underlying discrete-time model is such that the growth of log consumption follows a stationary random walk, so that this model is a special case of the one studied in Section 3. We assume in this section that the per-period change in log consumption compounds two normal distributions:

$$g\sim(h_1,1-p;h_2,p)\qquad\text{with}\quad h_i\sim N(\mu_i,\sigma_i^2), \tag{34}$$

The log consumption growth compounds a "business-as-usual" random variable $h_1\sim N(\mu_1,\sigma_1^2)$ with probability $1-p$, with a catastrophe event $h_2\sim N(\mu_2,\sigma_2^2)$ with probability $p$ and $\mu_2<<0<\mu_1$ and $\sigma_2\geq\sigma_1$. Barro (2006, 2009) convincingly explains that the risk-free puzzle and the equity premium puzzle can be explained by using credible values of the intensity $\mu_2$ of the macro catastrophe and of its frequency $p$. However, Martin (2013) shows that the levels of the $r_{f0}$ and $\pi_0(1)$ are highly sensitive to the frequency $p$ of rare events, and that this parameter $p$ is extremely difficult to estimate. In this section, in parallel to Collin-Dufresne et al. (2016), we contribute to this emerging literature by integrating this source of parametric uncertainty into the asset pricing model.

In the absence of parametric uncertainty, Eqs. (11) and (12) imply that the term structures of risk-adjusted discount rates are flat. We hereafter assume alternatively that $p$ is uncertain. Because a change in $p$ affects all cumulants of $g|p$ simultaneously, these cumulants are not independent. We calibrate this model as in the Discounted Expected Utility benchmark version of Barro (2006, 2009) and Martin (2013). We assume that $\delta=3\%$ and $\gamma=4$. In the business-as-usual scenario, the trend of growth is $\mu_1=2.5\%$ and its volatility is equal to $\sigma_1=2\%$. In case of a catastrophe, the trend of growth is $\mu_2=-39\%$ and the volatility is $\sigma_2=25\%$. Finally, we take a Bayesian approach for the estimation of the probability of a catastrophe. Suppose that our prior belief about $p$ is uninformative. Barro (2006) observes 60 catastrophes over 3500 country-years. Using Bayes' rule, it is well-known that this yields that the posterior distribution for $p$ is $Beta(61,3441)$.[8] The posterior mean of $p$ is 1.74%, and its posterior standard deviation is 0.22%.[9]

In Table 1, we report the discount rates for different maturities and asset's beta. As said before, this model is good to fit observed asset prices with short maturities. Moreover, as noticed by Collin-Dufresne et al. (2016), the learning process about $p$ is very slow, so that parameter uncertainty has a limited impact for small maturities. However in this calibration, we also obtain an interest rate of $-2.6\%$ for a 200-year maturity. The uncertain frequency of catastrophes also makes the term structure of risk premia increasing. But as for the risk-free rates, the sensitivity of risk premia to maturities is quite small, with an increase from 6.0% for a one-year maturity to 8.2% for a 200-year maturity. Notice that interest rates decrease faster than the speed at which risk premia increase, so that the discount rates for consumption claims decrease. These could explain why the price of leaseholds with an expiration time between 50 and 200 years are sold with a relatively large discount compared to freeholds, as shown by Giglio et al. (2015) and Giglio et al. (2015). This is compatible with our findings if real estate assets are not riskier than consumption claims.

[8] See Leamer (1978) for example.

[9] Collin-Dufresne et al. (2015) also consider the case of an uncertain probability of catastrophic events. Their calibration is different because they use Epstein-Zin preferences, and the conditional log consumption process is not a random walk. They show that the uncertainty affecting the transition probabilities has a large effect on the expected excess return of equity.

**Table 1**
The annual interest rate $r_{ft}$ , the systematic risk premia $\pi_t(\beta=1)$ and the discount rate for consumption claims when the probability of macroeconomic catastrophe is uncertain, with $p \sim Beta(61, 3441)$.

| | $t$ =1 year | $t$ =10 years | $t$ =200 years | $t \to \infty$ |
|---|---|---|---|---|
| $r_{ft}$ | 0.2% | 0.1% | −2.6% | −203.0% |
| $\pi_t(1)$ | 6.0% | 6.1% | 8.2% | 63.4% |
| $\rho_t(1)$ | 6.2% | 6.2% | 5.6% | −139.6% |

Table 1 also documents the asymptotic values of the interest rate and of the systematic risk premia, which are equal respectively to $r_{f\infty}=-203\%$ and $\pi_\infty(1)=63.4\%$ per annum. These numbers demonstrates that the calibration proposed by Barro (2006) is useful to explain the price of short-lived assets, but is potentially problematic to predict the interest rate and risk premium for extra-long maturities once the uncertainty affecting the probability of catastrophes is recognized. These results are in line with a recent paper by Collin-Dufresne et al. (2016) who consider a two-regime Markov model, with one rare regime yielding a catastrophically negative trend. They show that among the 6 parameters that govern this stochastic process, the uncertainty about the probability of a catastrophe has a much bigger impact on asset prices than the uncertainty affecting the conditional growth trends and the conditional volatilities. Their calibration leads to a conditional equity premium of 22%.

## 5. Application 2: Brownian growth process with an unknown trend or volatility

In this section, we examine the special case of our general model in which log consumption follows an arithmetic Brownian motion with unknown constant drift $\mu$ and volatility $\sigma$. Let us first consider the case in which the trend is known, but the volatility is ambiguous. Weitzman (2007) examined this question by assuming in addition that $\sigma^2$ has an inverted Gamma distribution. This implies that the unconditional growth rate $g$ has a Student's t-distribution rather than a normal, yielding fat tails, an unboundedly low safe discount rate and an unboundedly large market risk premium. In our terminology, because the Inverse Gamma distribution for $\sigma^2$ has no real CGF, Eqs. (11) and (12) offer another proof of Weitzman (2007)'s inexistence result. If we relax Weitzman's assumption governing our beliefs on the unknown volatility of growth, applying Lemma 1 to Eqs. (11) and (12) yields

$$r_{ft}=\delta+\gamma\mu-\frac{1}{2}\gamma^2\kappa_1^{\sigma^2}-\frac{1}{8}\gamma^4\kappa_2^{\sigma^2}t-\frac{1}{48}\gamma^6\kappa_3^{\sigma^2}t^2-\ldots \tag{35}$$

$$\pi_t(\beta)=\beta\gamma\kappa_1^{\sigma^2}+\frac{1}{2}\beta\gamma\kappa_2^{\sigma^2}\left[\beta^2-\frac{3}{2}\beta\gamma+\gamma^2\right]t+\ldots \tag{36}$$

In the remainder of this section, we alternatively follow Veronesi (2000) by considering the symmetric case in which only the drift of growth is uncertain.[10] Suppose first that the current beliefs about the true value of the drift can be represented by a normal distribution: $\mu \sim N(m_1^\mu, m_2^\mu)$. Applying property *ii* recursively in each double-CGF operations appearing in Eqs. (11) and (12) implies the following characterization:

$$r_{ft}=r_{f0}-0.5\gamma^2m_2^\mu t \tag{37}$$

$$\pi_t(\beta)=\beta\gamma\left(\sigma^2+m_2^\mu t\right) \tag{38}$$

Eq. (38) can be seen as an application of Eq. (10) since $G_t$ is Gaussian in this case with $Var(G_t)=\sigma^2t+m_2^\mu t^2$. In this Gaussian world, the standard use of filtering techniques allows for the analytical characterization of the time-varying term structures of the risk-adjusted discount rates.

In the remainder of this section, we explore the case of non-Gaussian distributions to describe our current beliefs about the trend of growth. Applying Lemma 1 to the equations in Theorem 1 yields:

$$\rho_t(\beta)=\rho_0(\beta)+t^{-1}\chi(t,\beta(\mu-m_1^\mu))-t^{-1}\chi(t,(\beta-\gamma)(\mu-m_1^\mu)). \tag{39}$$

Observe that if $\mu$ has a symmetric distribution, Eq. (39) implies that $\rho_t(\beta=0.5\gamma)$ is constant in $t$. Moreover, because $\chi(t,kx)=\chi(tk,x)$ for all $k \in \mathbb{R}$, the above equation implies that

$$\frac{\partial\rho_t(\beta)}{\partial t\partial\gamma}=(\beta-\gamma)\frac{\partial^2\chi}{\partial t^2}(t,(\beta-\gamma)(\mu-m_1^\mu)). \tag{40}$$

[10] Collin-Dufresne et al. (2016) also examine this case with a normally distributed prior for the growth trend. Their numerical analysis in the DEU case confirms that the instantaneous risk-free rate is not affected by the parameter uncertainty, and that the slope of the term structure of risk-free rate is decreasing. Stronger results are obtained in an Epstein-Zin framework, with the additional feature that the short risk-free rate is negatively impacted by parameter uncertainty under that framework.

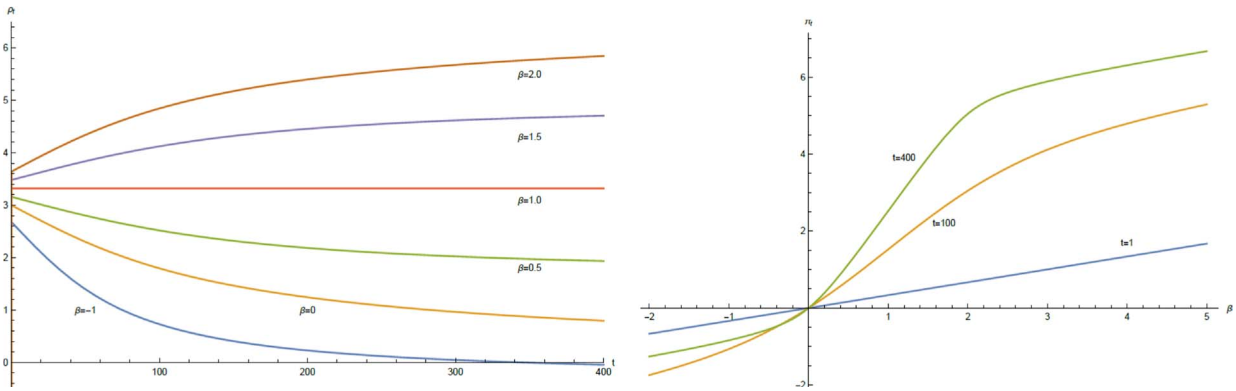


**Fig. 1.** The discount rate as a function of maturity (left) and beta (right). We assume that $\delta=0$, $\gamma=2$, $\sigma=4\%$ and $\mu$ is $U[0\%,3\%]$. Rates are in %.

Because $\chi$ is convex in its first argument, this implies that $\partial\rho_t(\beta)/\partial t$ is decreasing in $\gamma$ if and only if $\beta$ is smaller than $\gamma$. Combining these observations, we obtain the following result.

**Proposition 6.** : *Suppose that log consumption follows an arithmetic Brownian motion with a known volatility $\sigma\in\mathbb{R}^+$ and with an unknown trend $\mu$ with a bounded support. When $\mu$ is symmetrically distributed, the term structure of the discount rates is flat when $\beta=\gamma/2$. The term structure of the risk-adjusted discount rates is decreasing when $\beta$ is smaller than $\gamma/2$, and it is increasing when $\beta$ is in interval $[\gamma/2,\gamma]$.*

This result is stronger than the one described in Corollary 1 because it is not restricted to small maturities. We now characterize the asymptotic properties of the term structure of discount rates when the distribution of the trend $\mu$ has a bounded support $[\mu_{\min},\mu_{\max}]$. We first rewrite condition (39) as

$$\rho_t(\beta)=\delta+\gamma\sigma^2(\beta-0.5\gamma)+t^{-1}\chi(t,\beta\mu)-t^{-1}\chi(t,(\beta-\gamma)\mu). \tag{41}$$

Property $v$ of Lemma 1 tells us that $a^{-1}\chi(a,x)$ tends to the supremum of the support of $x$ when $a$ tends to infinity. If $\beta$ is negative, the supremum of the support of $\beta\mu$ is $\beta\mu_{\min}$, and the supremum of the support of $(\beta-\gamma)\mu$ is $(\beta-\gamma)\mu_{\min}$. This implies that the sum of the last two terms of the RHS of the above equality tends to $\gamma\mu_{\min}$. The other two cases are characterized in the same way to finally obtain the following result:

$$\rho_\infty(\beta)=\begin{cases}\delta+\gamma\sigma^2(\beta-0.5\gamma)+\gamma\mu_{\min} & \text{if}\quad \beta\le 0\\ \delta+\gamma\sigma^2(\beta-0.5\gamma)+(\gamma-\beta)\mu_{\min}+\beta\mu_{\max} & \text{if}\quad 0<\beta\le\gamma\\ \delta+\gamma\sigma^2(\beta-0.5\gamma)+\gamma\mu_{\max} & \text{if}\quad \beta>\gamma\end{cases} \tag{42}$$

For distant futures, the ambiguity affecting the trend is crucial for the determination of the discount rate. The long term wealth effect is equal to the product of $\gamma$ by a growth rate of consumption belonging to its support $[\mu_{\min},\mu_{\max}]$. Its selection depends here upon the beta of the project. When $\beta$ is negative, the wealth effect should be computed on the basis of the smallest possible growth rate $\mu_{\min}$ of the economy. On the contrary, when $\beta$ is larger than $\gamma$, the wealth effect should be computed on the basis of the largest possible rate $\mu_{\max}$.

Eq. (42) also tells us that the condition of a symmetric distribution for $\mu$ in Proposition 4 cannot be relaxed. Indeed, Eq. (42) implies that $\rho_0(\beta=\gamma/2)$ and $\rho_\infty(\beta=\gamma/2)$ are equal only if $E\mu=m_1^\mu$ and $(\mu_{\min}+\mu_{\max})/2$ coincide. Most asymmetric distributions will not satisfy this condition, which implies that the constancy of $\rho_t(\beta=\gamma/2)$ with respect to $t$ will be violated.

In Fig. 1, we illustrate some of the above findings in the following numerical example. We assume that $\delta=0$, $\gamma=2$, $\sigma=4\%$ and $\mu$ is uniformly distributed on interval $[0\%,3\%]$. As predicted by Proposition 6, the term structure is decreasing, flat or increasing depending upon whether $\beta$ is smaller, equal, or larger than $\gamma/2=1$. Observe also that, contrary to the Gaussian case, risk premia are not proportional to $\beta$. For example, consider a time horizon of 400 years. For this maturity, the risk premia associated to $\beta=1$ and $\beta=4$ are respectively equal to $\pi_{400}(1)=2.5\%$ and $\pi_{400}(4)=6.3\%<4\pi_{400}(1)$.

In this example, the sensitiveness of the discount rate to changes in the beta, which measures the local risk premium, is increasing in the maturity of the cash flows, in line with the idea that the term structure of risk premia is increasing. The aggregate risk premium is increased tenfold from $\pi_0(1)=0.3\%$ to $\pi_\infty(1)=3.3\%$ when maturity goes from 0 to infinity. This shows that it is particularly crucial to estimate the beta of projects having long-term impacts on the economy. Projects whose main benefits are to reduce emissions of greenhouse gases illustrate this point. It is thus particularly disappointing that we know virtually nothing about the “climate beta”. Second, this observation should reinforce the need to promote the integration of risk measures in the evaluation of public policies with long-term impacts. Up to our knowledge (Gollier, 2011), France is the only OECD country in which public institutions are requested to estimate the beta of the public investment projects.[11]

## 6. A model with persistent shocks to growth

In the benchmark specification with a CRRA utility function and a Brownian motion for log consumption, the term structures of risk-adjusted discount rates are flat and constant through time. In the specification with some parametric uncertainty on this Brownian motion that we examined in the previous section, they are monotone with respect to maturity and they move smoothly through time due to the revision of beliefs about the true values of the uncertain parameters. But this specification ignores the cyclicality of the economic activity. The introduction of predictable changes in the trend of growth introduces a new ingredient to the evaluation of investments. When expectations are temporarily diminishing, the discount rate associated to short horizons should be reduced to bias investment decisions toward projects that dampen the forthcoming temporary recession. Long termism is a luxury that should be favored only in periods of economic prosperity with relatively pessimistic expectations for the distant future.

In this section, we propose a simple model in which the economic growth is cyclical, with some uncertainty about the parameter governing this process. Following Bansal and Yaron (2004) and Hansen et al. (2008) for example, we assume in this section that the change in log consumption follows an auto-regressive process: [12]

$$\begin{aligned} \ln c_{t+1}/c_t &= x_{t+1} \\ x_{t+1} &= \mu_\theta + y_t + \varepsilon_{xt+1}, \\ y_{t+1} &= \phi y_t + \varepsilon_{yt+1}, \end{aligned} \tag{43}$$

for some initial (potentially ambiguous) state characterized by $y_0$, where $\varepsilon_{xt}$and$\varepsilon_{yt}$are independent and serially independent with mean zero and variance $\sigma_x^2$ and $\sigma_y^2$, respectively. Parameter $\phi$, which is between 0 and 1, represents the degree of persistence in the expected growth rate process. We hereafter allow the trend of growth $\mu_\theta$ to be uncertain. By forward induction of (43), it follows that:

$$G_t = \ln c_t/c_0 = \mu_\theta t + y_0 \frac{1-\phi^t}{1-\phi} + \sum_{\tau=1}^{t-1} \frac{1-\phi^{t-\tau}}{1-\phi} \varepsilon_{y\tau} + \sum_{\tau=1}^{t} \varepsilon_{x\tau}. \tag{44}$$

It implies that, conditional on $\theta$, $G_t$ is normally distributed with annualized variance

$$v_t = t^{-1} Var(G_t) = \frac{\sigma_y^2}{t(1-\phi)^2}\left[t-1-2\phi\frac{1-\phi^{t-1}}{(1-\phi)} + \phi^2 \frac{1-\phi^{2t-2}}{(1-\phi^2)}\right] + \sigma_x^2. \tag{45}$$

Using property *ii* of Lemma 1, Eqs. (8) and (9) imply that

$$\begin{aligned} \rho_t(\beta) &= \delta + t^{-1}(\chi(\beta, G_t) - \chi(\beta-\gamma, G_t)) \\ &= \delta + \gamma y_0 \frac{1-\phi^t}{t(1-\phi)} + \gamma(\beta - 0.5\gamma) v_t + t^{-1}(\chi(t, \beta\mu) - \chi(t, (\beta-\gamma)\mu)), \end{aligned} \tag{46}$$

where the annualized variance $v_t$of log consumption is given by Eq. (45). One can then treat the last term in the RHS of this equation as in the previous section. Bansal and Yaron (2004) consider the following calibration of the model, using annual growth data for the United States over the period 1929–1998. Taking the month as the time unit, they obtained,$\mu = 0.0015$, $\sigma_x = 0.0078$, $\sigma_y = 0.00034$, and $\phi = 0.979$. Using this $\phi$ yields a half-life for macroeconomic shocks of 32 months. Let us assume that $\delta = 0$, and let us introduce some uncertainty about the historical trend of growth from the sure $\mu = 0.0015$ to the uncertain context with two equally likely trends $\mu_1 = 0.0005$ and $\mu_2 = 0.0025$. In Fig. 2, we draw the term structures of discount rates for three different positions $y_0$ in the business cycle. In the set of curves starting from below, the current growth rate of the economy is 0.6% per year, well below its unconditional expectation of 1.8%. In this recession phase, the short term discount rate is a low 1%, but the expectation of a recovery makes the term structure steeply increasing for low maturities. For low betas, the term structure is non-monotone because of the fact that for very distant maturities, the effect of parametric uncertainty eventually dominates. In the set of curves starting from above, the current growth rate is 1.2% per year above its unconditional expectation. In this expansionary phase of the cycle, the short term discount rate is large at around 6%, but is steeply decreasing for short maturities because of the diminishing expectations. At the middle of the business cycle, when expectations are in line with the historical average ($y_0 = 0$), the term structure corresponds to the set

[11] In the U.S., the Office of Management and Budget (OMB) recommend to use a flat discount rate of 7% since 1992. It was argued that the "7% is an estimate of the average before-tax rate of return to private capital in the U.S. economy" (OMB, 2003). In 2003, the OMB also recommended the use of a discount rate of 3%, in addition to the 7% mentioned above as a sensitivity. The 3% corresponds to the average real rate of return of the relatively safe 10-year Treasury notes. Interestingly enough, the recommended use of 3% and 7% is not differentiated by the nature of the underlying risk, and is independent of the time horizon of the project.

[12] A more general model entails a time-varying volatility of growth as in Bansal and Yaron (2004) and Hansen (2012). Mean-reversion in volatility is useful to explain the cyclicality of the market risk premium.

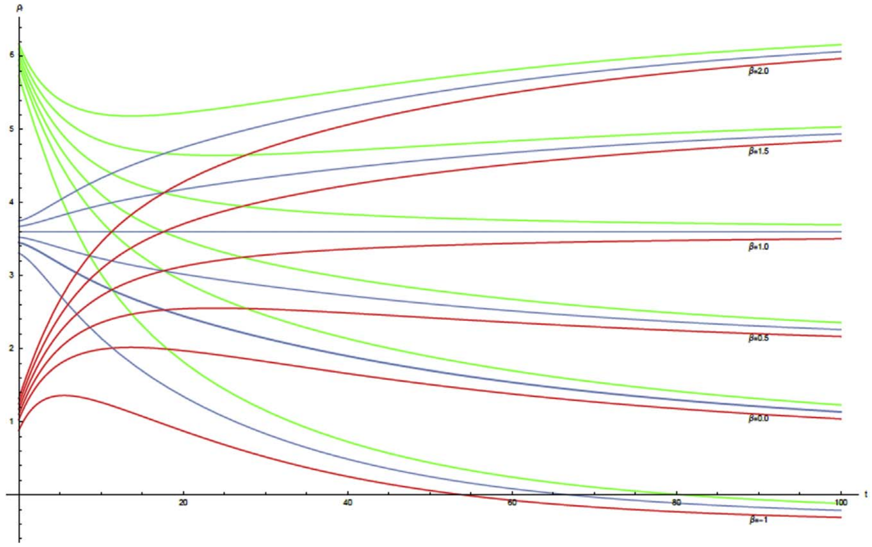


**Fig. 2.** The discount rate (in % per year) as a function of the maturity (in years) for different betas and for different states of the economy. Eq. (46) is calibrated with $\delta=0$, $\gamma=2$,$\sigma_x=0.0078$, $\sigma_y=0.00034$, $\phi=0.979$, two equally likely trends $\mu_1=0.0005$ and $\mu_2=0.0025$, and $y_0=-0.001$ (bottom), $y_0=0$ (center), and for $y_0=0.001$ (top).

of curves starting from the center of the vertical axis. Although short-term discount rates are highly sensitive to the position in the business cycle, the long-term discount rates are not.

## 7. Why is the term structure of equity decreasing?

Our goal in this paper is normative. We showed that the same ingredient, i.e., uncertainty about the parameters of the random walk of consumption growth rates, justifies using both a downward-sloping term structure of risk-free discount rates and an upward-sloping term structure of systematic risk premia. This is true in particular when the risks affecting the different cumulants of the growth rate are statistically independent. However, we show in the Appendix that our model may generate a locally decreasing term structure for risk premia when the uncertain cumulants are correlated. We show in the Appendix that this may be the case in particular when the trend and the volatility of aggregate consumption growth are negatively correlated.

The correlation of the uncertain cumulants may thus be a potential explanation for the observation that the term structure of equity premia is empirically downward-sloping. This fact has first been shown for maturities up to 10 years by van Binsbergen et al. (2012, 2013) and van Binsbergen and Koijen (2016). More recently, Giglio et al. (2015a, 2015b) showed that the rates at which housing cash-flows are discounted have a decreasing term structure for maturities ranging from 50 to 999 years. They estimated housing discount rates by comparing real estate prices of freeholds (with infinite property rights) to those of leaseholds (with property rights of fixed maturity from 50 years to 999 years), both in the UK and in Singapore. The discount on the price of leaseholds with respect to freeholds reflects the present value – and the equilibrium long-term discount rate – of the perpetual flow of housing benefits after the leasehold expiry. The relatively high discount observed on these markets suggests that the risk-adjusted housing discount rates are low for very long maturities, implying for example a discount rate below 2.6% for 100-year housing benefits. The downward-sloping term structure of discount rates is also compatible with the value premium, i.e., the observation that value stocks have a larger average risk premium than growth stocks, in spite of the fact that their betas are similar.

Following Eq. (10), this decreasing term structure of risk premia can be explained in the standard consumption-based CAPM only if the annualized variance of log consumption has a decreasing term structure. This requires some form of negative serial correlation or mean-reversion in consumption growth rates. To illustrate this point, we modify the auto-regressive model (43) to allow for the transitory shock on growth $x$ to be negatively correlated with shocks to its persistent component $y$:

$$\begin{aligned} &\ln c_{t+1}/c_t = x_{t+1} \\ &\quad x_{t+1} = \mu_\theta + y_t + \varepsilon_{xt+1}, \\ &\quad y_{t+1} = \phi y_t - \lambda \varepsilon_{xt+1} + \varepsilon_{yt+1}, \end{aligned} \tag{47}$$

When the new parameter $\lambda$ is positive, a positive transitory shock to growth reduces the future expected growth rate of consumption, thereby introducing a source of negative serial correlation that can potentially counterbalance the effect of the persistence of shocks to make the term structure of annualized volatility $v_t$ locally decreasing. Taking care of this extension

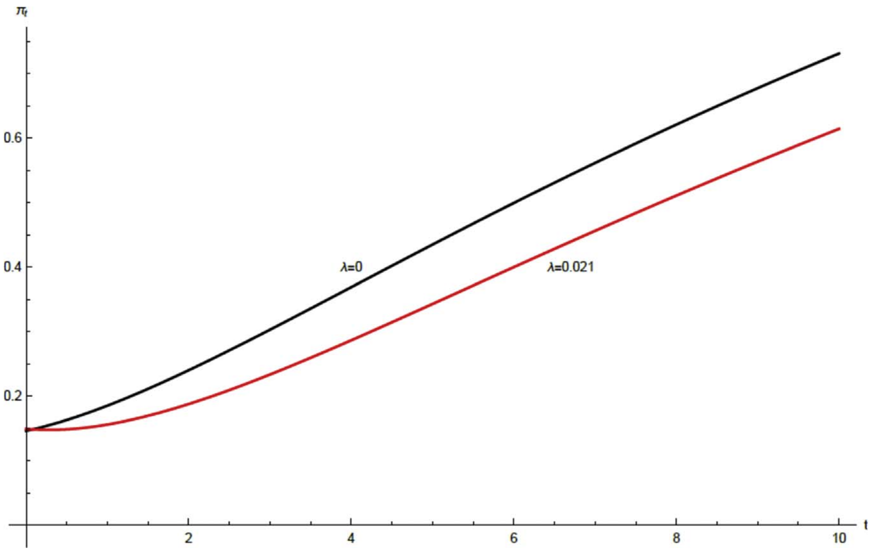


**Fig. 3.** The term structure of systematic risk premia (in % per year) when the transitory and persistent shocks to growth are negatively correlated. The top curve corresponds to the calibration used for Fig. 2, assuming $y_0=0$ and independent shocks. The other curve corresponds to the dynamic process described by Eq. (47) with a correlation parameter $\lambda=0.021$.

in the framework of Section 6 is straightforward. It yields the same pricing formula (46) in which the annualized volatility $v_t$ is now given by the following expression:

$$\begin{aligned} v_t = & \frac{\sigma_y^2}{t(1-\phi)^2}\left[t-1-2\phi\frac{1-\phi^{t-1}}{(1-\phi)}+\phi^2\frac{1-\phi^{2t-2}}{(1-\phi^2)}\right] \\ & +\frac{\sigma_x^2}{t(1-\phi)^2}\left[(t-1)(1-\phi-\lambda)^2+(1-\phi)^2+2(1-\phi-\lambda)\lambda\phi\frac{1-\phi^{t-1}}{(1-\phi)}+\lambda^2\phi^2\frac{1-\phi^{2t-2}}{(1-\phi^2)}\right]. \end{aligned} \tag{48}$$

It is easy to verify that the asymptotic annualized variance is minimized by selecting $\lambda=1-\phi$. This is a case in which the second term of the above expression vanishes completely for long maturities, i.e., the negative correlation eliminates the impact of short-term volatility on long-run risks. Thus, $\lambda=1-\phi$ minimizes the term spread of equity. Is this effect large enough to dominate the effect of parametric uncertainty? To examine this question, we reexamine the calibration of the Bansal-Yaron model described in Fig. 2, in which $\sigma_x=0.0078$ and $\phi=0.979$. Taking $\lambda=1-\phi=0.021$ rather than $\lambda=0$ as in Fig. 2 reduces the long-term annualized volatility $\sqrt{v_\infty}$ from 1.81% to 1.63%. This reduces the asymptotic risk premium from 3.19% to 3.04%. It is largely insufficient to compensate the effect of parametric uncertainty, as shown in Fig. 3.

Thus, this form of negative serial correlation can help reducing the spread on equity returns, but it can solve the puzzle of a negative term structure of equity only if one ignores parameter uncertainty. Lettau and Wachter (2007) obtained a downward-sloping term structure of equity in a model in which dividends follow a stochastic process with negative serial correlation equivalent to system (47). But because their pricing kernel is not consumption-based, it is not easy to relate their findings to our results. Nakamura et al. (2013) introduced a negative serial correlation in growth rates by allowing the economy to potentially recover after a disaster. In the absence of parametric uncertainty, this unsurprisingly yields a downward-sloping term structure of equity.

In the long-run risk model of Bansal and Yaron (2004), the representative agent is assumed to have full information about the level of the persistent component of consumption growth. Croce et al. (2015) observe that this is an unrealistic assumption of their model, because this component must be small compared to the transitory component. They show that a decreasing term structure of equity can be obtained as an equilibrium in this model if the representative agent has imperfect information about the current state of the economy. To explore this question, let us reexamine the simple Bansal-Yaron's long-run risk model (43) in which we now assume that the persistent component $y_t$ is unobservable but is subject to learning through the observation of future consumption growth. This implies that the current state $y_0$ – rather than the trend $\mu$ as in the previous section – is uncertain. In the expected utility framework, we obtain that the risk-adjusted discount rate is given by the following equation:

$$\begin{aligned} \rho_t(\beta) = & \delta+\gamma\mu+\gamma(\beta-0.5\gamma)v_t \\ & +t^{-1}\left(\chi\left(1,\beta y_0\left(1-\phi^t\right)/(1-\phi)\right)-\chi\left(1,(\beta-\gamma)y_0\left(1-\phi^t\right)/(1-\phi)\right)\right), \end{aligned} \tag{49}$$

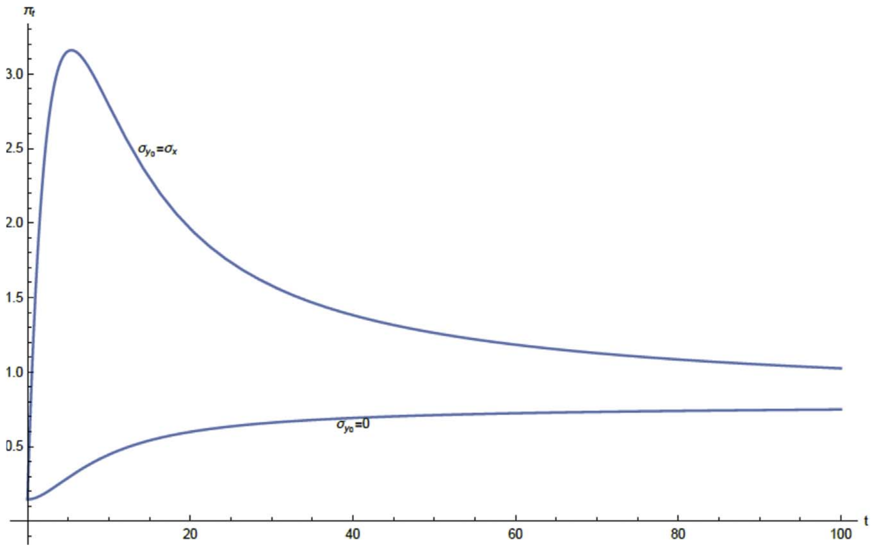


**Fig. 4.** The term structure of systematic risk premia (in % per year) under imperfect information. The calibration of the dynamic system (43) is with $\delta=0$, $\gamma=2$,$\sigma_x=0.0078$, $\sigma_y=0.00034$, $\phi=0.979$, and $\mu=0.0015$, as in Bansal and Yaron (2004). The lower curve corresponds to the case with perfect information ($y_0=0$ with probability 1), whereas the other curve is with imperfect information ($Ey_0=0$ and $\sigma_{y_0}=0.0078$).

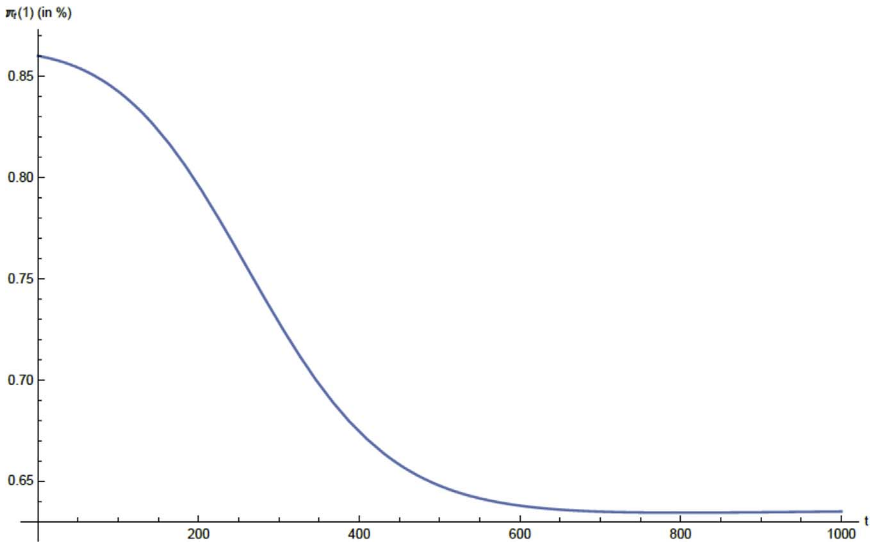


**Fig. 5.** Counterexample in which risk premia are locally decreasing in maturity.

where the annualized variance $v_t$ is given by Eq. (45). For simplicity, let us also assume that the beliefs about the current state of the economy is given by $y_0 \sim N\left(0,\sigma_{y_0}^2\right)$. In that case, the above equation can be rewritten as

$$\rho_t(\beta)=\delta+\gamma\mu+\gamma(\beta-0.5\gamma)\left[v_t+\left(\frac{1-\phi^t}{1-\phi}\right)^2\frac{\sigma_{y_0}^2}{t}\right]. \tag{50}$$

This yields the following risk premia:

$$\pi_t(1)=\rho_t(1)-\rho_t(0)=\gamma v_t+\gamma\left(\frac{1-\phi^t}{1-\phi}\right)^2\frac{\sigma_{y_0}^2}{t}. \tag{51}$$

The first term in the right-hand side of this equality is the traditional upward-sloping term structure of annualized variance coming from the persistence of shocks to the growth rate of consumption. The second term comes from the inability to observe the level of the persistent component. It has a hump-shaped term structure, for the following reasons. For small maturities, the uncertainty affecting the state variable generates positive correlation in growth rates, thereby magnifying long-run risks. But contrary to the uncertainty affecting the permanent trend $\mu$, the uncertainty affecting the level of the current state of the economy has an evanescent influence on the uncertainty affecting long-term growth. This implies that the term structure of equity is hump-shaped if the uncertainty affecting the persistent component of growth is large enough. This is illustrated in Fig. 4. In this figure, we consider a dynamic process (43) with the calibration of Bansal and Yaron (2004) in which we add an uncertain $y_0$ with a zero mean and a standard deviation equaling the volatility $\sigma_x$ of the transitory shock to the growth rate of consumption. This analysis shows that imperfect information in the consumption-based CAPM can explain the decreasing nature of the term structure of equity only for medium and long maturities.

Another possible avenue to solve the puzzle is to extend the model to Epstein-Zin-Weil preferences, as is now the norm in asset pricing theory. But Croce et al. (2015) show that the term structure of equity remains upward-sloping under perfect information with long-run risks. Gollier and Kihlstrom (2016) show that when the representative agent has preferences for an early resolution of uncertainty, the anticipation of an early resolution of macroeconomic uncertainty tends to make the term structure of equity less upward-sloping. So, Epstein-Zin-Weil preferences can only be a partial answer to the puzzle, as also explained also by Giglio et al. (2015) and van Binsbergen and Koijen (2016).

## 8. Concluding remarks

Our main messages in this framework are as follows. First, investors and policy evaluators should be particularly concerned by parameter uncertainty when they value dividends and benefits that materialize in the distant future. Although parameter uncertainty has *no* effect on the value of very short term zero-coupon bonds and consumption claims, it has an increasingly important impact on risk-adjusted discount rates when contemplating longer maturities.

The shape of the term structure of risk-adjusted discount rates for risky projects is determined by the relative intensity of a precautionary effect that makes the term structure of the risk-free discount rates decreasing, and of a risk aversion effect that tends to make the term structure of risk premia increasing, under some conditions. Our results are generic in the sense that they do not depend upon whether the source of the deep uncertainty is about the trend or volatility of growth, the frequency or the intensity of catastrophes, or any other parameter that affect the distribution of the per-period growth rate of consumption in the economy.

Our results are normative, but the findings that the term structure of systematic risk premia is increasing is in conflict with the recent observation that equity premia observed on markets have a decreasing term structure. Some explanations for this disagreement have been provided in the paper, but the debate is still open. We insist that in the normatively appealing expected utility model, any persistence in shocks to the growth rate of the economy magnifies the long run risk, and thereby should make interest rates and risk premia respectively decreasing and increasing with maturities, because of respectively prudence and risk aversion. Deep parameter uncertainty and learning generate this persistence in shocks to conditional growth rates, thereby yielding the same policy recommendations. This is why we have supported a regulatory reform on this matter in France and elsewhere. Public investment projects in France are now evaluated by using a maturity-sensitive risk-adjusted discount rate $\rho_t = r_{ft} + \beta\pi_t$, where $\beta$ is the consumption-based CAPM beta of the project, $r_{ft}$ is the decreasing risk-free rate and $\pi_t$ is the increasing systematic risk premium.[13] Given the large efficient asymptotic risk premium that emerges from the calibration of different models with realistic parameter uncertainties, the standard recommendation existing in many Western countries to use a single discount rate to evaluate public policies independent of their risk profile is particularly problematic for policies having long-lasting socioeconomic impacts.[14]

## Appendix. On the possibility of an increasing term structure of aggregate risk premia

We have shown in Proposition 3 that the term structure of aggregate risk premia must be increasing when the set of uncertain cumulants of the increments of log consumption are statistically independent. Eq. (23) shows that this result may not be robust to correlated cumulants. In this appendix, we build a counterexample. Take $\beta = 1$ and $\gamma = 0.1$. Suppose that the log consumption process is Brownian with unknown trend and volatility. There are two states of nature. The first state $\theta = 1$ yields a zero trend and a high volatility: $\mu_1 = 0\%$ and $\sigma_1 = 30\%$. The second state $\theta = 2$ yields a higher trend and a lower volatility: $\mu_2 = 5\%$ and $\sigma_2 = 10\%$. The probability of state 1 is $p_1 = 0.95$.

In Fig. 5, we draw the term structure of aggregate risk premia in this economy. Observe that the aggregate risk premium is locally decreasing in maturity.

[13] See Quinet (2013). The risk-free rate goes from 2.5% for maturities smaller than 50 years down to 1.5% for longer maturities. The risk premium goes from 1.5% to 3%.

[14] Notice that some public investments are so large than they have a non-marginal impact on the aggregate consumption growth. In that case, the marginalist approach used in asset-pricing theory can be used only as an approximation of the true value of the project. It is recommended to use a general equilibrium approach in that case.

This counterexample illustrates the fact that the cumulants of order larger than 2 also affect the aggregate risk premium. To see this, we can rewrite Eq. (9) as:

$$\pi_t(\beta)=\sum_{n=2}^{\infty}\frac{\kappa_n^{\ln c_t}}{t}\frac{\beta^n-(\beta-\gamma)^n+(-\gamma)^n}{n!}. \tag{52}$$

For $\beta=1$ and $\gamma=0.1$, this simplifies to

$$\pi_t(1)=0.1\frac{\kappa_2^{\ln c_t}}{t}+0.045\frac{\kappa_3^{\ln c_t}}{t}+0.0143\frac{\kappa_4^{\ln c_t}}{t}+\ldots \tag{53}$$

Observe that, under these values of the parameters, the risk premium is increasing in the first three annualized cumulants of log consumption. After some tedious computations, we also obtain that

$$\begin{aligned} t^{-1}\kappa_2^{\ln c_t} &= E\kappa_2^{g|\theta}+\kappa_2^{\mu|\theta}t \\ &= 8.6\times 10^{-2}+1.1875\times 10^{-4}t \end{aligned} \tag{54}$$

$$\begin{aligned} t^{-1}\kappa_3^{\ln c_t} &= E\kappa_3^{g|\theta}+3\mathrm{cov}\left(\kappa_1^{g|\theta},\kappa_2^{g|\theta}\right)t+\kappa_3^{\mu|\theta}t^2 \\ &= -5.7\times 10^{-4}t+5.34375\times 10^{-6}t^2 \end{aligned} \tag{55}$$

$$\begin{aligned} t^{-1}\kappa_4^{\ln c_t} &= 3tVar(\kappa_2^{g|\theta})+6\mathrm{cov}(\kappa_2^{g|\theta},(\mu_\theta-\mu)^2)t^2+\kappa_4^{\mu|\theta}t^3 \\ &= 9.12\times 10^{-4}t-5.13\times 10^{-5}t^2+2.12\times 10^{-7}t^3 \end{aligned} \tag{56}$$

Observe from Eqs. (54)–(56) that for small maturities, the term structures of the annualized variance and of the annualized excess kurtosis are increasing in maturity $t$, whereas the term structure of the annualized skewness is decreasing. It happens in this example that the dominating effect for the slope of the term structure of risk premia comes from the decreasing skewness. This means that the RHS of Eq. (23) is negative.

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